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OPTIMUM AIRPLANE FLIGHT PATHS*

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SUMMARY

The generalized equations are discussed which pertain to an airplane executing its flight path in a single vertical plane, under a sequence of stipulations specifying the nature of the local optima which are distinctly different in character along successive portions of this path. The aerodynamic and propulsive characteristics of the airplane are allowed to be specified with complete generality.

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INTRODUCTION

In a previous paper[1] the equations were discussed which govern the least-time optimization of the flight path of a given airplane. It is the purpose of this present study to extend the concepts utilized there in order to make them apply to a much broader classification of such problems, including now the question of minimizing the fuel consumption. It is proposed to examine the necessary and sufficient conditions which must be fulfilled in order to give assurance that the desired optima will be achieved throughout the consecutive stretches of the connected trajectory along which the airplane is supposed to be operating according to the dictates of the several individual and different restrictions imposed successively on its behavior. It will be noted that the boundary conditions are assigned in such a way as to preserve complete generality in the treatment being expounded.

Particular attention is to be focused on the transition process which takes place in passing over, from one scallop of the trajectory where the airplane is obeying the dictates of a particular set of conditions, to come abruptly under the influence of a distinctly different set of imposed constraints on an adjacent portion of its path. Detailed analyses will serve to illustrate how a quite general method of attack may be set up for arriving at the sought solution to such flight paths, specified by constraining conditions that can change in character from one adjoining scallop of trajectory to the next.

No attempt will be made here to recapitulate all the basic hypotheses upon which these analyses have been predicated; the interested reader is referred for these particulars to the exposition given in Reference 1. Likewise the bibliographic material listed in that reference should also be consulted for purposes of better orientation. Nevertheless, it is appropriate to make special mention here of the important paper by Garfinkel[2] wherein he presents a truly penetrating variational treatment of airplane performance, although, unfortunately, his study is partially restricted because of some of the confining hypotheses imposed; in particular, there is a limitation to small angles of inclination for the flight path, and, in addition, a special sort of engine is premised.

DEFINITIONS AND NOTATIONS

Let the mass of the airplane be denoted by m. This mass is considered to be concentrated at a point; the velocity vector $\vec{V}$ which is associated with the motion of this point is agreed to lie entirely within a single vertical (x, z)-plane. The following forces are assumed to be applied to the mass in question: the weight $W = m\,g$ is aligned with the negatively directed vertical ordinate; i. e., it acts along $-\vec{z}$, while the lift, L, is taken to be perpendicular to $\vec{V}$. Likewise, the drag, D, and the thrust, T, are taken to be parallel to the vector $\vec{V}$. The angle that the tangent to the trajectory makes with the horizontal is to be represented by the symbol ϑ, so that ϑ is the angle between $\vec{V}$ and the x-axis.

Furthermore let it be premised that the drag and the lift are going to be known functions of the speed, V, the altitude, z, and the angle of attack, α, which determines the orientation of the airplane with respect to the direction of $\vec{V}$. Likewise let it be supposed that the thrust is given as a function of the speed, V, the altitude, z, and the fuel consumption, β, which is identical to the rate of decrease in the mass, m.

Let the customary convention also be adopted that a dot over a symbol is to denote that the total derivative of the so-indicated quantity with respect to time is meant. In addition, the partial derivative of a quantity with respect to a certain variable will be denoted by affixing the corresponding subscript to the symbol representing the function in question, with the exception that the subscripts i and j, or any numeral, will have their usual ordering significance. If the ordering indices i or j appear twice as a subscript in any term, it is to be taken for granted, as is familiar from tensor notation, that a summation symbol stands in front of the term. Thus this shorthand notation indicates that the sum must be taken of all such terms with every sequential integer supplied throughout the range of the indicated indices.

In order to recast the present problem pertaining to flight path optimization into the more convenient notation appropriate to a general variational analysis, the following transliterations may be made:

Take $\varphi_1 = \dot{x} - V \cos \vartheta$ (1)

$\varphi_2 = \dot{z} - V \sin \vartheta$ (2)

$\varphi_3 = \dot{V} + g \sin \vartheta - (D-T)/m$ (3)

$\varphi_4 = V\dot{\vartheta} + g \cos \vartheta - L/m$ (4)

$\varphi_5 = \dot{m} + \beta$ (5)

$$\left.\begin{array}{l} y_0 = T,\ y_2 = z,\ y_4 = \vartheta \\ y_1 = x,\ y_3 = V,\ y_5 = m \end{array}\right\} \quad (6)$$

$F = \lambda_j \ \varphi_j$ (for $j = 1, \cdots, 5$) (7)

and $H = \dot{y}_i \ F_{\dot{y}_i}$ (for $i = 1, \cdots, 5$) (8)

where λ (t) are the Lagrangian multipliers.

The general variational problem under consideration may be stated now as follows, conforming to Mayer's interpretation: one wishes to find functions $y_1, \cdots y_5$, α, and β, depending on the temporal variable t, which will minimize the quantity that depends on the values that these variables take on at the end-points A and B of a specified interval. The quantity so to be minimized is usually expressed in the form of the difference produced in a certain given function G when evaluated at the ends of the selected interval AB; i.e., the function to be minimized is regularly written as G(B) - G(A). This is the meaning to be attached herein to the requirement for finding an optimum path. If one wishes to attempt the maximization of a functional quantity, it is resolved into nothing more than finding a minimum, as defined above, through mere reversal of the sign.

Now let the variation in the function G be denoted by δ G(A), so that

$$\delta G = G_{y_j} \ \delta y_j \ \text{(for } j = 0, \cdots, 5) \quad (9)$$

where the indicated derivatives of G are to be evaluated at the end-point A. Likewise, let a specialized variation in F be denoted by $\delta^*F(A)$, by which is meant that

$$\delta^* F = F_{\dot{y}_i} \ \delta y_i + H \, \delta t \quad \text{(for } i = 1, \cdots, 5) \quad (10)$$

where again the indicated derivatives of F are to be evaluated at the end-point A.

NECESSARY CONDITIONS FOR A MINIMUM

For solution of the variational problem according to the Mayer interpretation it is seen that the pertinent Euler's Equations, $dF_{\dot{y}_i}/dt = F_{y_i}$, for i taking the value of 1, 2, 3, and 5, now may be written explicitly as follows:

$$\dot{\lambda}_1 = 0 \tag{11}$$

$$\dot{\lambda}_2 + \lambda_3 (T_z - D_z)/m + \lambda_4 L_z/m = 0 \tag{12}$$

$$\dot{\lambda}_3 + \lambda_1 \cos\vartheta + \lambda_2 \sin\vartheta + \lambda_3 (T_V - D_V)/m + \lambda_4 (L_V/m - \dot{\vartheta}) = 0 \tag{13}$$

$$\dot{\lambda}_5 + \lambda_3 (D-T)/m^2 - \lambda_4 L/m^2 = 0 \tag{14}$$

In place of the corresponding equation obtained for the index i = 4 it will be most convenient to make use of the first integral for F, which is H = c where c is a constant. That this is a first integral for F may be readily recognized by the fact that from Eq. (7) one has $\partial F/\partial t = 0$. Consequently, the constraining relation that exists between the variables in this instance is

$$\lambda_1 \dot{x} + \lambda_2 \dot{z} + \lambda_3 \dot{V} + \lambda_4 V \dot{\vartheta} + \lambda_5 \dot{m} = c \tag{15}$$

In the case of the variables α and β the corresponding Euler's Equations are simply $F_\alpha = F_\beta = 0$. If then the differential quotient D_α/L_α is written simply as D_L, it follows that the corresponding restraints involving α and β are

$$\lambda_4 = \lambda_3 D_L \tag{16}$$

and $$\lambda_5 = \lambda_3 T_\beta/m \tag{17}$$

Thus, the set of equations written down here as Eqs. (11) through (17) constitutes, when used in conjunction with the equilibrium conditions and definitions embodied in setting $\varphi_i = 0$, for i running from 1 through 5, a differential system, which must be satisfied by the several variables involved when they are describing the sought optimum trajectory.

If it should so happen that one wants to invoke the condition that for a part of the trajectory there shall be no change in altitude, so that z is constant, then it will be necessary to relinquish the constraint embodied in Eq. (12). Likewise, if the angle of attack is not allowed to vary, so that α remains constant, then it is necessary to relinquish the condition prescribed by Eq. (16). In similar fashion, if the fuel consumption is considered to be fixed - for example, when the engine is kept at full throttle - then it will be necessary to drop Eq. (17).

The boundary conditions to be imposed are 12 in number. These boundary conditions are made up of a certain number of assigned stipulations, but the rest will be imposed, of course, in consequence of the equivalence expressed as

$$\delta^* F(A) + \delta G(A) = \delta^* F(B) + \delta G(B) \tag{18}$$

which must hold regardless of the arbitrary choices that are made for the variations, δy, to which the differential system may be subjected, provided merely that the constraints are not violated by such choices. It is not possible to discuss the existence of an actual solution for a variational problem thus set, except by consideration of specific cases.

For a particular example, let the situation be considered wherein the end-point A is prescribed, or let it be presumed that the values of $y_0, \cdots y_5$ are assigned to start with. For such situations the significance of Eq. (18) becomes specialized to the statement that the right hand side of the equation must equal zero. Then, if it is demanded that the duration of the flight is to be a minimum, one must set $c = 1$, while if it is required that the farthest x-value be attained, or the highest z-value or the fastest V-value, it would be necessary to impose the condition that $\lambda_1 = 1$, or $\lambda_2 = 1$, or $\lambda_3 = 1$*, respectively, at the end-point B.

If nothing is specified about the time of flight, or if the time of flight does not explicitly appear in the function G nor in the statement of the boundary conditions, then one takes $c = 0$. If the horizontal distance flown is unrestricted, then λ_1 is to be taken equal to zero. If, on the other hand, the angle, ϑ, of the flight path with respect to the horizontal is allowed to have any value at will by the time the end-point B is reached, then in these circumstances, the Lagrangian nultiplier $\lambda_4 = 0$. Thus, whenever the angle of attack is not selected beforehand, then the angle of attack is required to vary according to the dictates of Eq. (16), which means that the plane has to fly in the attitude for minimum drag.

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*Nothing is altered if one selects some value other than unity when invoking these conditions. It may be noted, in fact, that Eqs. (11) through (17) are homogeneous in the Lagrangian multipliers, and, besides, these multipliers do not enter into the statement of the imposed physical boundary conditions. Furthermore, it is obvious that the solution which minimizes G will also provide the minimum for the case when any positive factor multiplies this G-function.

In the event that one does not choose to take into account any variation in the weight of the airplane ($F_m = 0$), it is necessary to have $\dot{\lambda}_5 = 0$. Under such an assumption and provided there is no overriding restriction on the way the fuel consumption has to behave, then the rate of fuel consumption has to be selected as the value giving $T_\beta = 0$, or else it has to remain fixed at one of its end-point values.

Finally, if it is required to solve the interception problem, which consists of selecting the best route to follow in order to collide with another airplane having the general path specified by $x = X(t)$

$$z = Z(t)$$

it is necessary to invoke the stipulation (the so-called intercept condition) that

$$\lambda_1 \dot{X} + \lambda_2 \dot{Z} = \text{constant} \quad . \tag{19}$$

SUFFICIENT CONDITIONS FOR A MINIMUM

The strong conditions of Legendre for a minimum demand that one should have

$$F_{\dot{y}_i \dot{y}_j} \delta\dot{y}_i \delta\dot{y}_j > 0 \qquad (\text{for } i,\ j = 1, \cdots, 7) \tag{20}$$

for all variations $\delta\dot{y}$ that obey the conditions

$$\varphi_{j\,\dot{y}_i} \delta\dot{y}_i = 0 \qquad \begin{array}{l}(\text{for } j = 1, \cdots, 5)\\ (\text{and for } i = 1, \cdots, 7)\end{array} \tag{20'}$$

where the further notational simplification has been introduced of setting

$$\dot{y}_6 = \alpha$$

$$\text{and} \quad \dot{y}_7 = \beta$$

Upon carrying out the indicated operations, it will be observed that Eq. (20) implies that it is sufficient for assuring a minimum that one should have $F_{\alpha\alpha} > 0$ and $F_{\beta\beta} > 0$. The first of these conditions may be more explicitly written as

$$\lambda_3 (D_{\alpha\alpha} - D_L L_{\alpha\alpha})/m > 0 \tag{21}$$

If the lift and drag can be written in the customary separated form suggested by dimensional analysis, so that

$$L = c'(\alpha) f(V, z) \quad \text{and} \quad D = c''(\alpha) f(V, z) \tag{22}$$

where c' and c" stand for the usual coefficients of lift and drag, then the sufficient condition written as Eq. (21) reduces just to the requirement that

$$\lambda_3 \frac{d^2c''}{dc'^2} > 0 \tag{23}$$

In consequence of invoking the other sufficiency condition, one obtains the explicit expression

$$\lambda_3 T_{\beta\beta} < 0 \tag{24}$$

The appropriate Weierstrass condition which applies in this instance may be written as

$$E = \lambda_3 (\Delta D - D_L \Delta L - \Delta T + T_\beta \Delta\beta) \geq 0 \tag{25}$$

where the significance of the symbols is as follows. The ΔL and ΔD represent the variations produced in L and in D, respectively, in consequence of an arbitrary change in the angle of attack, $\Delta\alpha$. Likewise ΔT stands for the variation in thrust produced by an arbitrary change $\Delta\beta$, when V and z are held fixed. The condition written as Eq. (25) constitutes an extension, now applying over the entire operating range of the airplane or engine, to the requirement for concavity or convexity that the Legendre sufficiency conditions impose merely in the neighborhood of a single operating point on the airplane polar and the engine thrust chart.

The conventional conditions for a minimum, as stated by Jacobi, may be reformulated in various ways in order to extend their original concept and mode of application[3]. These applications will require different analytic developments, however, depending on the kinds of boundary condition which enter the specific problem in hand. Because of the complexity of the required developments and because it must be acknowledged that in practice one has to deal with many extremal solutions, the problem usually resolves itself into a check merely on whether or not the extremals in question are going to cover simply, in the mathematical sense, that part of the plane of the trajectory plots where the solution of interest lies.

DISCONTINUITIES IN THE AIRPLANE'S POLAR OR IN THE ENGINE'S THRUST

Now consider what happens at a transition point between two scallops of the sought trajectory where two different kinds of restraint are to be obeyed on either side of this point of demarcation. At such a point the Erdmann-Weierstrass conditions may be satisfied by setting equal the corresponding values of the special variation δ^*F. This has to be done at each instance where there occurs a discontinuity in the polar curve of the lifting ensemble or in the thrust curve of the engine. If the inequalities written as Eq. (20) are satisfied on both sides of the point of transition, then the imposition of the above-mentioned conditions leads to the conclusion that the Lagrangian multipliers and the constant c ought not to change value in traversing the transition point in question.

Let attention be focused, in particular, upon what transpires at a location where there is a sharp change in the value of D_L. It will be seen that in this case the continuity of λ_3 and λ_4 is not compatible with the restraint stated as Eq. (16). Therefore, this constraint has to be disregarded, and one proceeds along a piece of extremal curve for which α remains constant. By proceeding along this scallop of the trajectory one finally reaches the point where the two Lagrangian multipliers take on such values as to once again satisfy the constraint embodied in Eq. (16) even with the new jumped-in-value slope D_L. After this happens, then the angle of attack α once more begins to vary in just the same way it did before, or else in the opposite sense, depending on whether the ratio λ_4/λ_3 has already attained the value that D_L takes on along the new branch of the polar D(L) or whether it still has the value belonging to the old branch.

Quite similar considerations apply for analogous corner-points (points of discontinuity in the derivative) encountered on the thrust curve for the engine, $T(\beta)$. In this instance it becomes necessary to disregard the constraint written as Eq. (17) and to take β = constant over a stretch of the trajectory. Proceeding along this scallop of the trajectory, holding β = constant, one continues on until he attains the location where the value of λ_5/λ_3 becomes such as to satisfy once again Eq. (17), either on the old branch or on the new branch of the operating curve, which in general depends on V and z.

It should be remarked that the optimal trajectory can sustain instantaneous jumps in the angle of incidence or in the discharge rate of the propellant in certain cases even when the airplane polar and engine thrust curves behave in a continuous way, and have continuous derivatives besides. Such a possibility has to be taken into account when one is confronted with a singular problem, in particular when one has to deal with the situation wherein a portion or several portions of the polar D(L), or the engine thrust curve $T(\beta)$, is made up of a piece or pieces of straight line.

This particular difficulty also comes about indirectly when the polar and thrust curves are such as to not satisfy the Legendre and Weierstrass conditions everywhere. Take, for example, the situation where the lift and drag are defined as in Eq. (22), but where it so happens, as is commonly held to be the case for laminar "low-drag" type airfoils, that there is a bucket in the drag polar $c''(c')$ of such a nature that a tangent can meet the curve in two separate and distinct points A and B.

Arriving at the point A along one branch of such a polar the convexity condition $d^2c''/dc'^2 > 0$ will continually be in force, and then one may immediately make the transition to point B, where the jumps in the non-dimensional coefficients $\Delta c'$ and $\Delta c''$ corresponding to the segment AB will have for their ratio a value equal to the slope at A, just as is demanded for satisfaction of the Erdmann-Weierstrass conditions, as stated in Eq. (15). Mathematically it would be possible, likewise, for one to traverse the AB segment of the trajectory by means of a sequence of instantaneous increments in the angle of incidence, as one goes from one extremity, A, to the other extremity, B, on the polar in question. Of course, such a solution is not compatible with physical reality because of the inertial resistance to rotation which a real airplane would have, but which is being ignored in this treatment. The straight line connections thus serve to lop off the convolutions of portions of the polar where the Weierstrass conditions would not be satisfied. Thus the polar in question will take on the appearance of having a stretched string joining any crests or troughs that may be present inherently. Quite similar observations* apply to the thrust curve $T(\beta)$.

THE SIMPLIFIED PROBLEM

If one is content to ignore the centrifugal forces, then it is clear that $\dot{\vartheta}$ will drop out of Eqs. (4) and (13), and consequently the job of determining the extremals can be carried out in quite simple fashion. The Legendre condition will not, however, be satisfied. In this case, in fact, because Eqs. (1) through (8) do not contain $\dot{\vartheta}$, it is necessary to assume that $\dot{y}_4 = \vartheta$. Then one finds that the equation, corresponding to Eq. (21), which now applies in these circumstances will contain the factor

$$(D_{\alpha\alpha} - D_L L_{\alpha\alpha}) \sin^2\vartheta - D_L L_\alpha^2 / L$$

which can become negative, as may be verified by making use of Eq. (22). When this is done it is seen that the above-given expression may be rewritten as follows, save possibly for positive multipliers:

$$\sin^2\vartheta \frac{d^2c''}{dc'^2} - \frac{dc''}{c'dc'}$$

and this is always a negative quantity when the polar is parabolic.

Consider, for example, the case of horizontal free-flight ($\lambda_1 = 0$) and the thrust held constant (β = constant). In this instance it is found that the "trajectory" for minimum time is composed of pieces of straight line, finite or infinitesimal in length, along which the attitude of the plane is verticle ($\vartheta = \pm \frac{\pi}{2}$). To better appreciate this situation, observe that when the problem is being analyzed by the direct approach one has to admit that to achieve the shortest-time path a maneuver has to be executed that is composed of a sequence of fragments of trajectory, at the beginning and end of which the plane is in a vertical dive, while during the central portion of which the plane zooms up in a climb, so that the following relation holds

$$V(T_V - D_V + L_V D_L) + T - D = V^2 (T_z - D_z + L_z D_L)/g.$$

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*If E is a maximum or minimum point on the thrust curve $T(\beta)$, representing maximum or minimum discharge rate of the propellant, and if close by to this extremum one has that $T_{\beta\beta} > 0$, it is not necessary to leave the curve at the point of inflection, as Garfinkel states in Reference 2, in order to obey the Legendre conditions, but rather one must leave the thrust curve at the point where the tangent passes through E, in order to conform to the Weierstrass condition (so long as the tangent passes above the origin where $T = \beta = 0$).

Thus the complete path is made up of a continuous succession of such wild excursions between the limit values of the plane's attitude.

In contrast to this physically absurd solution one may note that a sensible result is provided by use of the indirect method, illustrated in more detail* in Reference 4. Admittedly, the indirect solution also does not satisfy the conditions of Legendre and Weierstrass, but it is illuminating in that it gives the sought trajectory as a series of aperiodic zooms, or, that is, by means of paths along which there is no dependence on the attitude ϑ, and yet this sort of solution comes very close to the exact solution which is obtained when one takes into account the actual centrifugal forces.

CONCLUSIONS

The problem that has been given some scrutiny here in its general form will not present any real difficulty when one takes up the formulation of the problem for various specific sub-cases. Neither does it appear that the numerical applications should be excessively laborious, provided that the boundary conditions apply entirely to one of the end points of the flight path, because in this case the solution may be worked out by use of a step-by-step integration. The most serious obstacle in the way of applying this method in practice lies in the fact that the boundary conditions will be required to be met at both end points of the path, in most engineering problems of interest. In such a situation it will be found to be rather tiresome to try to pick out the correct initial values by trial and error, so that after working through the calculations one ends up with the desired trajectory, especially in view of the fact that the determination of numerous extremal curves will always be found necessary in order to make sure that the Jacobi conditions are not contradicted.

The numerical applications can be carried out with ease, however, if the procedure is adopted of making up the desired extremals out of a series of trajectory scallops, along each of which α and β are held constant. When this is done it is found that the problem may be reduced to a question of finding just an ordinary

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*See in particular Eqs. (10) and (11) of the cited reference.

minimum. A calculation along this line of approach made with the equations presented above lends itself much more logically, however, to an analysis employing the technique of successive approximations.

In using the scallops of trajectory along which α and β are held constant, it must be recognized that Eqs. (16) and (17) are not going to be in force. These equations may be used, however, to reveal the instant when one must transfer from one branch to the other. The transfer from one scallop of trajectory to the next should be made as soon as

$$\lambda_4/\lambda_3 = \Delta D/\Delta L$$

or when

$$\lambda_5 m/\lambda_3 = \Delta T/\Delta\beta$$

where the symbol Δ is used to indicate the jump corresponding to the selected interval over which α and β are going to vary while V and z are held fixed.

If at first, the calculations are carried out by use of only a limited number of selected trim positions of the airplane and with a small number of throttle settings, then a more refined analysis may be made later by taking into account a larger number of such selected values. This process of better approximations may be carried out in a systematic way, so as to serve as a continuous check on the flight paths so determined.

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